Two remarks on composition operators on the Dirichlet space

Daniel Li, Hervé Queffélec, Luis Rodriguez-Piazza

Two remarks on composition operators on the Dirichlet space

Daniel Li, Hervé Queffélec,
Luis Rodríguez-Piazza[*]

July 10, 2014

Abstract. *We show that the decay of approximation numbers of compact composition operators on the Dirichlet space $\mathcal{D}$ can be as slow as we wish. We also prove the optimality of a result of O. El-Fallah, K. Kellay, M. Shabankhah and H. Youssfi on boundedness on $\mathcal{D}$ of self-maps of the disk all of whose powers are norm-bounded in $\mathcal{D}$.*

Mathematics Subject Classification. Primary: 47B33 – Secondary: 46E22; 47B06 ; 47B32

Key-words. approximation numbers – Carleson embedding – composition operator – cusp map – Dirichlet space

1 Introduction

Recall that if φ is an analytic self-map of $\mathbb{D}$, a so-called *Schur function*, the composition operator C_φ associated to φ is formally defined by

$$C_\varphi(f) = f \circ \varphi\,.$$

The Littlewood subordination principle ([4], p. 30) tells us that C_φ maps the Hardy space H^2 to itself for every Schur function φ. Also recall that if H is a Hilbert space and $T\colon H \to H$ a bounded linear operator, the n-th approximation number $a_n(T)$ of T is defined as

$$a_n(T) = \inf\{\|T - R\|\,;\ \operatorname{rank} R < n\}, \quad n = 1, 2, \ldots\,. \tag{1.1}$$

In [12], working on that Hardy space H^2 (and also on some weighted Bergman spaces), we have undertaken the study of approximation numbers $a_n(C_\varphi)$ of composition operators C_φ, and proved among other facts the following:

[*]Supported by a Spanish research project MTM 2012-05622.

Theorem 1.1 *Let $(\varepsilon_n)_{n\geq 1}$ be a non-increasing sequence of positive numbers tending to* 0*. Then, there exists a compact composition operator C_φ on H^2 such that*

$$\liminf_{n\to\infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0\,.$$

As a consequence, there are composition operators on H^2 which are compact but in no Schatten class.

The last item had been previously proved by Carroll and Cowen ([3]), the above statement with approximation numbers being more precise.

For the Dirichlet space, the situation is more delicate because not every analytic self-map of $\mathbb{D}$ generates a bounded composition operator on $\mathcal{D}$. When this is the case, we will say that φ is a *symbol* (understanding "of $\mathcal{D}$"). Note that every symbol is necessarily in $\mathcal{D}$.

In [11], we have performed a similar study on that Dirichlet space $\mathcal{D}$, and established several results on approximation numbers in that new setting, in particular the existence of symbols φ for which C_φ is compact without being in any Schatten class S_p. But we have not been able in [11] to prove a full analogue of Theorem 1.1. Using a new approach, essentially based on Carleson embeddings and the Schur test, we are now able to prove that analogue.

Theorem 1.2 *For every sequence $(\varepsilon_n)_{n\geq 1}$ of positive numbers tending to* 0*, there exists a compact composition operator C_φ on the Dirichlet space $\mathcal{D}$ such that*

$$\liminf_{n\to\infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0\,.$$

Turning now to the question of necessary or sufficient conditions for a Schur function φ to be a symbol, we can observe that, since $(z^n/\sqrt{n})_{n\geq 1}$ is an orthonormal sequence in $\mathcal{D}$ and since formally $C_\varphi(z^n) = \varphi^n$, a necessary condition is as follows:

$$\varphi \text{ is a symbol} \quad\Longrightarrow\quad \|\varphi^n\|_{\mathcal{D}} = O\,(\sqrt{n})\,. \tag{1.2}$$

It is worth noting that, for any Schur function, one has:

$$\varphi\in\mathcal{D} \quad\Longrightarrow\quad \|\varphi^n\|_{\mathcal{D}} = O\,(n)$$

(of course, this is an equivalence). Indeed, anticipating on the next section, we have for any integer $n\geq 1$:

$$\begin{aligned}\|\varphi^n\|_{\mathcal{D}}^2 &= |\varphi(0)|^{2n} + \int_{\mathbb{D}} n^2\,|\varphi(z)|^{2(n-1)}|\varphi'(z)|^2\,dA(z)\\ &\leq |\varphi(0)|^2 + \int_{\mathbb{D}} n^2\,|\varphi'(z)|^2\,dA(z) \leq n^2\|\varphi\|_{\mathcal{D}}^2,\end{aligned}$$

giving the result.

Now, the following sufficient condition was given in [5]:

$$\|\varphi^n\|_{\mathcal{D}} = O\,(1) \quad \Longrightarrow \quad \varphi \text{ is a symbol}\,. \tag{1.3}$$

In view of (1.2), one might think of improving this condition, but it turns out to be optimal, as says the second main result of that paper.

Theorem 1.3 *Let $(M_n)_{n\geq 1}$ be an arbitrary sequence of positive numbers tending to ∞. Then, there exists a Schur function $\varphi \in \mathcal{D}$ such that:*

1) $\|\varphi^n\|_{\mathcal{D}} = O\,(M_n)$ as $n \to \infty$;

2) φ is not a symbol on $\mathcal{D}$.

The organization of that paper will be as follows: in Section 2, we give the notation and background. In Section 3, we prove Theorem 1.2; in Section 3.1, we prove Theorem 1.3; and we end with a section of remarks and questions.

2 Notation and background.

We denote by $\mathbb{D}$ the open unit disk of the complex plane and by A the normalized area measure $dx\,dy/\pi$ of $\mathbb{D}$. The unit circle is denoted by $\mathbb{T} = \partial\mathbb{D}$. The notation $A \lesssim B$ indicates that $A \leq c\,B$ for some positive constant c.

A Schur function is an analytic self-map of $\mathbb{D}$ and the associated composition operator is defined, formally, by $C_\varphi(f) = f\circ\varphi$. The operator C_φ maps the space $\mathcal{H}ol\,(\mathbb{D})$ of holomorphic functions on $\mathbb{D}$ into itself.

The Dirichlet space $\mathcal{D}$ is the space of analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{\mathcal{D}}^2 := |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2\, dA(z) < +\infty\,. \tag{2.1}$$

If $f(z) = \sum_{n=0}^{\infty} c_n z^n$, one has:

$$\|f\|_{\mathcal{D}}^2 = |c_0|^2 + \sum_{n=1}^{\infty} n\,|c_n|^2\,. \tag{2.2}$$

Then $\|\,.\,\|_{\mathcal{D}}$ is a norm on $\mathcal{D}$, making $\mathcal{D}$ a Hilbert space, and $\|\,.\,\|_{H^2} \leq \|\,.\,\|_{\mathcal{D}}$. For further information on the Dirichlet space, the reader may see [1] or [16].

The Bergman space $\mathfrak{B}$ is the space of analytic functions $f\colon \mathbb{D} \to \mathbb{C}$ such that:

$$\|f\|_{\mathfrak{B}}^2 := \int_{\mathbb{D}} |f(z)|^2\, dA(z) < +\infty\,.$$

If $f(z) = \sum_{n=0}^{\infty} c_n z^n$, one has $\|f\|_{\mathfrak{B}}^2 = \sum_{n=0}^{\infty} \frac{|c_n|^2}{n+1}$. If $f \in \mathcal{D}$, one has by definition:

$$\|f\|_{\mathcal{D}}^2 = \|f'\|_{\mathfrak{B}}^2 + |f(0)|^2\,.$$

Recall that, whereas every Schur function φ generates a bounded composition operator C_φ on Hardy and Bergman spaces, it is no longer the case for the Dirichlet space (see [14], Proposition 3.12, for instance).

We denote by $b_n(T)$ the n-th *Bernstein number* of the operator $T\colon H \to H$, namely:

$$b_n(T) = \sup_{\dim E = n} \Big(\inf_{f \in S_E} \|Tx\| \Big) \tag{2.3}$$

where S_E denotes the unit sphere of E. It is easy to see ([11]) that

$$b_n(T) = a_n(T) \quad \text{for all } n \geq 1\,.$$

(recall that the approximation numbers are defined in (1.1)).

If φ is a Schur function, let

$$n_\varphi(w) = \#\{z \in \mathbb{D}\,;\ \varphi(z) = w\} \geq 0 \tag{2.4}$$

be the associated *counting function*. If $f \in \mathcal{D}$ and $g = f \circ \varphi$, the change of variable formula provides us with the useful following equation ([17], [11]):

$$\int_{\mathbb{D}} |g'(z)|^2 \, dA(z) = \int_{\mathbb{D}} |f'(w)|^2 \, n_\varphi(w) \, dA(w) \tag{2.5}$$

(the integrals might be infinite). In those terms, a necessary and sufficient condition for φ to be a symbol is as follows ([17], Theorem 1). Let:

$$\rho_\varphi(h) = \sup_{\xi \in \mathbb{T}} \int_{S(\xi,h)} n_\varphi \, dA \tag{2.6}$$

where $S(\xi, h) = \mathbb{D} \cap D(\xi, h)$ is the Carleson window centered at ξ and of size h. Then φ is a symbol if and only if:

$$\sup_{0<h<1} \frac{1}{h^2} \, \rho_\varphi(h) < \infty. \tag{2.7}$$

This is not difficult to prove. In view of (2.5), the boundedness of C_φ amounts to the existence of a constant C such that:

$$\int_{\mathbb{D}} |f'(w)|^2 \, n_\varphi(w) \, dA(w) \leq C \int_{\mathbb{D}} |f'(z)|^2 \, dA(z)\,, \quad \forall f \in \mathcal{D}.$$

Since $f' = h$ runs over $\mathfrak{B}$ as f runs over $\mathcal{D}$, and with equal norms, the above condition reads:

$$\int_{\mathbb{D}} |h(w)|^2 \, n_\varphi(w) \, dA(w) \leq C \int_{\mathbb{D}} |h(z)|^2 \, dA(z)\,, \quad \forall h \in \mathfrak{B}.$$

This exactly means that the measure $n_\varphi \, dA$ is a Carleson measure for $\mathfrak{B}$. Such measures have been characterized in [7] and that characterization gives (2.7).

But this condition is very abstract and difficult to test, and sometimes more "concrete" sufficient conditions are desirable. In [11], we proved that, even if the Schur function extends continuously to $\overline{\mathbb{D}}$, no Lipschitz condition of order α, $0 < \alpha < 1$, on φ is sufficient for ensuring that φ is a symbol. It is worth noting that the limiting case $\alpha = 1$, so restrictive it is, guarantees the result.

Proposition 2.1 *Suppose that the Schur function φ is in the analytic Lipschitz class on the unit disk, i.e. satisfies:*

$$|\varphi(z) - \varphi(w)| \leq C\,|z - w|\,, \quad \forall z, w \in \mathbb{D}\,.$$

Then C_φ is bounded on $\mathcal{D}$.

Proof. Let $f \in \mathcal{D}$; one has:

$$\begin{aligned}\|C_\varphi(f)\|_{\mathcal{D}}^2 &= |f\big(\varphi(0)\big)|^2 + \int_{\mathbb{D}} |f'\big(\varphi(z)\big)|^2 |\varphi'(z)|^2\, dA(z)\\ &\leq |f\big(\varphi(0)\big)|^2 + \|\varphi'\|_\infty^2 \int_{\mathbb{D}} |f'\big(\varphi(z)\big)|^2\, dA(z)\,.\end{aligned}$$

This integral is nothing but $\|C_\varphi(f')\|_{\mathfrak{B}}^2$ and hence, since C_φ is bounded on the Bergman space $\mathfrak{B}$, we have, for some constant K_1:

$$\int_{\mathbb{D}} |f'\big(\varphi(z)\big)|^2\, dA(z) \leq K_1^2 \|f'\|_{\mathfrak{B}}^2 \leq K_1^2 \|f\|_{\mathcal{D}}^2\,.$$

On the other hand,

$$|f\big(\varphi(0)\big)| \leq (1 - |\varphi(0)|^2)^{-1/2} \|f\|_{H^2} \leq (1 - |\varphi(0)|^2)^{-1/2} \|f\|_{\mathcal{D}}\,,$$

and we get

$$\|C_\varphi(f)\|_{\mathcal{D}}^2 \leq K^2 \|f\|_{\mathcal{D}}^2\,,$$

with $K^2 = K_1^2 + (1 - |\varphi(0)|^2)^{-1}$. □

3 Proof of Theorem 1.2

We are going to prove Theorem 1.2 mentioned in the Introduction, which we recall here.

Theorem 3.1 *For every sequence (ε_n) of positive numbers with limit 0, there exists a compact composition operator C_φ on $\mathcal{D}$ such that*

$$\liminf_{n\to\infty} \frac{a_n(C_\varphi)}{\varepsilon_n} > 0\,.$$

Before entering really in the proof, we may remark that, without loss of generality, by replacing ε_n with $\inf(2^{-8}, \sup_{k\geq n} \varepsilon_k)$, we can, and do, assume that $(\varepsilon_n)_n$ decreases and $\varepsilon_1 \leq 2^{-8}$.

Moreover, we can assume that $(\varepsilon_n)_n$ decreases "slowly", as said in the following lemma.

Lemma 3.2 *Let (ε_i) be a decreasing sequence with limit zero and let $0 < \rho < 1$. Then, there exists another sequence $(\widehat{\varepsilon_i})$, decreasing with limit zero, such that $\widehat{\varepsilon_i} \geq \varepsilon_i$ and $\widehat{\varepsilon_{i+1}} \geq \rho\,\widehat{\varepsilon_i}$, for every $i \geq 1$.*

Proof. We define inductively $\widehat{\varepsilon_i}$ by $\widehat{\varepsilon_1} = \varepsilon_1$ and

$$\widehat{\varepsilon_{i+1}} = \max(\rho\,\widehat{\varepsilon_i}, \varepsilon_{i+1}).$$

It is seen by induction that $\widehat{\varepsilon_i} \geq \varepsilon_i$ and that $\widehat{\varepsilon_i}$ decreases to a limit $a \geq 0$. If $\widehat{\varepsilon_i} = \varepsilon_i$ for infinitely many indices i, we have $a = 0$. In the opposite case, $\widehat{\varepsilon_{i+1}} = \rho\,\widehat{\varepsilon_i}$ from some index i_0 onwards, and again $a = 0$ since $\rho < 1$. $\square$

We will take $\rho = 1/2$ and assume for the sequel that $\varepsilon_{i+1} \geq \varepsilon_i/2$.

Proof of Theorem 3.1. We first construct a subdomain $\Omega = \Omega_\theta$ of $\mathbb{D}$ defined by a cuspidal inequality:

$$\Omega = \{z = x + iy \in \mathbb{D}\,;\ |y| < \theta(1-x)\,,\ 0 < x < 1\}\,, \tag{3.1}$$

where $\theta\colon [0,1] \to [0,1[$ is a continuous increasing function such that

$$\theta(0) = 0 \quad \text{and} \quad \theta(1-x) \leq 1 - x\,. \tag{3.2}$$

Note that since $1 - x \leq \sqrt{1-x^2}$, the condition $|y| < \theta(1-x)$ implies that $z = x + iy \in \mathbb{D}$. Note also that $1 \in \overline{\Omega}$ and that Ω is a Jordan domain.

We introduce a parameter δ with $\varepsilon_1 \leq \delta \leq 1 - \varepsilon_1$. We put:

$$\theta(\delta^j) = \varepsilon_j\,\delta^j \tag{3.3}$$

and we extend θ to an increasing continuous function from $(0,1)$ into itself (piecewise linearly, or more smoothly, as one wishes). We claim that:

$$\theta(h) \leq h \quad \text{and} \quad \theta(h) = o\,(h) \text{ as } h \to 0\,. \tag{3.4}$$

Indeed, if $\delta^{j+1} \leq h < \delta^j$, we have $\theta(h)/h \leq \theta(\delta^j)/\delta^{j+1} = \varepsilon_j/\delta$, which is $\leq \varepsilon_1/\delta \leq 1$ and which tends to 0 with h.

We define now $\varphi = \varphi_\theta\colon \overline{\mathbb{D}} \to \overline{\Omega}$ as a continuous map which is a Riemann map from $\mathbb{D}$ onto Ω, and with $\varphi(1) = 1$ (a cusp-type map). Since φ is univalent, one has $n_\varphi = \mathbb{1}_\Omega$, and since Ω is bounded, φ defines a symbol on $\mathcal{D}$, by (2.7). Moreover, (3.4) implies that $A[S(\xi,h) \cap \Omega] \leq h\,\theta(h)$ for every $\xi \in \mathbb{T}$; hence, ρ_φ being defined in (2.6), one has $\rho_\varphi(h) = o\,(h^2)$ as $h \to 0^+$. In view of [17], this little-oh condition guarantees the compactness of $C_\varphi\colon \mathcal{D} \to \mathcal{D}$.

It remains to minorate its approximation numbers.

The measure $\mu = n_\varphi\,dA$ is a Carleson measure for the Bergman space $\mathfrak{B}$, and it was proved in [10] that $C_\varphi^* C_\varphi$ is unitarily equivalent to the Toeplitz operator $T_\mu = I_\mu^* I_\mu\colon \mathfrak{B} \to \mathfrak{B}$ defined by:

$$T_\mu f(z) = \int_{\mathbb{D}} \frac{f(w)}{(1 - \overline{w}z)^2}\,dA(w) = \int_{\mathbb{D}} f(w) K_w(z)\,dA(w)\,, \tag{3.5}$$

where $I_\mu \colon \mathfrak{B} \to L^2(\mu)$ is the canonical inclusion and K_w the reproducing kernel of $\mathfrak{B}$ at w, i.e. $K_w(z) = \frac{1}{(1-\overline{w}z)^2}$.

Actually, we can get rid of the analyticity constraint in considering, instead of T_μ, the operator $S_\mu = I_\mu I_\mu^* \colon L^2(\mu) \to L^2(\mu)$, which corresponds to the arrows:

$$L^2(\mu) \xrightarrow{I_\mu^*} \mathfrak{B} \xrightarrow{I_\mu} L^2(\mu) \, .$$

We use the relation (3.5) which implies:

$$a_n(C_\varphi) = a_n(I_\mu) = a_n(I_\mu^*) = \sqrt{a_n(S_\mu)} \, . \tag{3.6}$$

We set:

$$c_j = 1 - 2\delta^j \quad \text{and} \quad r_j = \varepsilon_j \, \delta^j \tag{3.7}$$

One has $r_j = \varepsilon_j (1 - c_j)/2$.

Lemma 3.3 *The disks $\Delta_j = D(c_j, r_j)$, $j \geq 1$, are disjoint and contained in Ω.*

Proof. If $z = x + iy \in \Delta_j$, then $1 - x > 1 - c_j - r_j = (1 - c_j)(1 - \varepsilon_j/2) = 2\delta^j(1 - \varepsilon_j/2) \geq \delta^j$ and $|y| < r_j = \theta(\delta^j)$; hence $|y| < \theta(\delta^j) \leq \theta(1 - x)$ and $z \in \Omega$. On the other hand, $c_{j+1} - c_j = 2(\delta^j - \delta^{j+1}) = 2(1 - \delta)\delta^j \geq 2\varepsilon_1 \delta^j \geq 2\varepsilon_j \delta^j = 2r_j > r_j + r_{j+1}$; hence $\Delta_j \cap \Delta_{j+1} = \emptyset$. □

We will next need a description of S_μ.

Lemma 3.4 *For every $g \in L^2(\mu)$ and every $z \in \mathbb{D}$:*

$$I_\mu^* g(z) = \int_\Omega \frac{g(w)}{(1 - \overline{w}z)^2} \, dA(w) \tag{3.8}$$

$$S_\mu g(z) = \bigg(\int_\Omega \frac{g(w)}{(1 - \overline{w}z)^2} \, dA(w) \bigg) \, \mathbb{1}_\Omega(z) \, . \tag{3.9}$$

Proof. K_w being the reproducing kernel of $\mathfrak{B}$, we have for any pair of functions $f \in \mathfrak{B}$ and $g \in L^2(\mu)$:

$$\begin{aligned} \langle I_\mu^* g, f \rangle_{\mathfrak{B}} = \langle g, I_\mu f \rangle_{L^2(\mu)} &= \int_\Omega g(w) \overline{f(w)} \, dA(w) = \int_\Omega g(w) \, \langle K_w, f \rangle_{\mathfrak{B}} \, dA(w) \\ &= \Big\langle \int_\Omega g(w) K_w \, dA(w), f \Big\rangle_{\mathfrak{B}}, \end{aligned}$$

so that $I_\mu^* g = \int_\Omega g(w) K_w \, dA(w)$, giving the result. □

In the rest of the proof, we fix a positive integer n and put:

$$f_j = \frac{1}{r_j} \, \mathbb{1}_{\Delta_j} \, , \qquad j = 1, \ldots, n \, . \tag{3.10}$$

Let:

$$E = \operatorname{span} (f_1, \ldots, f_n) \, .$$

This is an n-dimensional subspace of $L^2(\mu)$.

The Δ_j's being disjoint, the sequence $(f_1, \ldots, f_n)$ is orthonormal in $L^2(\mu)$. Indeed, those functions have disjoint supports, so are orthogonal, and:

$$\int f_j^2 \, d\mu = \int f_j^2 \, n_\varphi \, dA = \int_{\Delta_j} \frac{1}{r_j^2} \, dA = 1 \, .$$

We now estimate from below the Bernstein numbers of I_μ^*. To that effect, we compute the scalar products $m_{i,j} = \langle I_\mu^*(f_i), I_\mu^*(f_j) \rangle$. One has:

$$\begin{aligned} m_{i,j} = \langle f_i, S_\mu(f_j) \rangle &= \int_\Omega f_i(z) \overline{S_\mu f_j(z)} \, dA(z) \\ &= \iint_{\Omega \times \Omega} \frac{f_i(z) \overline{f_j(w)}}{(1 - w\overline{z})^2} \, dA(z) \, dA(w) \\ &= \frac{1}{r_i r_j} \iint_{\Delta_i \times \Delta_j} \frac{1}{(1 - w\overline{z})^2} \, dA(z) \, dA(w) \, . \end{aligned}$$

Lemma 3.5 *We have*

$$m_{i,i} \geq \frac{\varepsilon_i^2}{32}, \qquad \textit{and} \quad |m_{i,j}| \leq \varepsilon_i \, \varepsilon_j \, \delta^{j-i} \quad \textit{for } i < j \, . \tag{3.11}$$

Proof. Set $\varepsilon_i' = \frac{r_i}{1 - c_i^2} = \frac{\varepsilon_i}{2(1 + c_i)}$. One has $\frac{\varepsilon_i}{4} \leq \varepsilon_i' \leq \frac{\varepsilon_i}{2}$. We observe that (recall that $A(\Delta_i) = r_i^2$):

$$m_{i,i} - {\varepsilon_i'}^2 = \frac{1}{r_i^2} \iint_{\Delta_i \times \Delta_i} \left[\frac{1}{(1 - w\overline{z})^2} - \frac{1}{(1 - c_i^2)^2} \right] dA(z) \, dA(w) \, .$$

Therefore, using the fact that, for $z \in \Delta_i$ and $w \in \mathbb{D}$:

$$|1 - w\overline{z}| \geq 1 - |z| \geq 1 - c_i - r_i = 1 - c_i - \varepsilon_i \Big(\frac{1 - c_i}{2} \Big) \geq (1 - c_i) \Big(1 - \frac{\varepsilon_i}{2} \Big) \geq \frac{1 - c_i}{2}$$

and then the mean-value theorem, we get:

$$\begin{aligned} |m_{i,i} - {\varepsilon_i'}^2| &\leq \frac{1}{r_i^2} \iint_{\Delta_i \times \Delta_i} \left| \frac{1}{(1 - w\overline{z})^2} - \frac{1}{(1 - c_i^2)^2} \right| dA(z) \, dA(w) \\ &\leq \frac{1}{r_i^2} \iint_{\Delta_i \times \Delta_i} \frac{32 \, r_i}{(1 - c_i)^3} \, dA(z) \, dA(w) \\ &= \frac{32 \, r_i^3}{(1 - c_i)^3} \leq 32 \times 8 \, {\varepsilon_i'}^3 \leq \frac{{\varepsilon_i'}^2}{2} \, , \end{aligned}$$

since $\varepsilon_i \leq \varepsilon_1 \leq 2^{-8}$ implies that $\varepsilon_i' \leq 1/(32 \times 16)$. This gives us the lower bound $m_{i,i} \geq {\varepsilon_i'}^2 / 2 \geq \varepsilon_i^2 / 32$.

Next, for $i < j$:

$$\begin{aligned} |m_{i,j}| &\leq \frac{1}{r_i r_j} \iint_{\Delta_i \times \Delta_j} \left| \frac{1}{(1 - w\overline{z})^2} \right| dA(z)\, dA(w) \leq \frac{1}{r_i r_j} \frac{4}{(1 - c_i)^2} r_i^2 r_j^2 \\ &= \frac{4\, \varepsilon_i\, \varepsilon_j\, \delta^{i+j}}{4\, \delta^{2i}} = \varepsilon_i\, \varepsilon_j\, \delta^{j-i}\,, \end{aligned}$$

and that ends the proof of Lemma 3.5. □

We further write the $n \times n$ matrix $M = (m_{i,j})_{1 \leq i,j \leq n}$ as $M = D + R$ where D is the diagonal matrix $m_i = m_{i,i}$ with $m_i \geq \frac{\varepsilon_i^2}{32}$, $1 \leq i \leq n$. Observe that M is nothing but the matrix of S_μ on the orthonormal basis $(f_1, \ldots, f_n)$ of E, so that we can identify M and S_μ on E.

Now the following lemma will end the proof of Theorem 3.1.

Lemma 3.6 *If $\delta \leq 1/200$, we have:*

$$\|D^{-1} R\| \leq 1/2\,. \tag{3.12}$$

Indeed, by the ideal property of Bernstein numbers, Neumann's lemma and the relations:

$$M = D(I + D^{-1}R)\,, \quad \text{and} \quad D = MQ \quad \text{with} \quad \|Q\| \leq 2,$$

we have $b_n(D) \leq b_n(M)\, \|Q\| \leq 2\, b_n(M)$, that is:

$$a_n(S_\mu) = b_n(S_\mu) \geq b_n(M) \geq \frac{b_n(D)}{2} = \frac{m_{n,n}}{2} \geq \frac{\varepsilon_n^2}{64}\,,$$

since the n first approximation numbers of the diagonal matrix D (the matrices being viewed as well as operators on the Hilbertian space $\mathbb{C}^n$ with its canonical basis) are $m_{1,1}, \ldots, m_{n,n}$. It follows that, using (3.6):

$$a_n(I_\mu) = a_n(I_\mu^*) = \sqrt{a_n(S_\mu)} \geq \frac{\varepsilon_n}{8}\,\cdot \tag{3.13}$$

In view of (3.6), we have as well $a_n(C_\varphi) \geq \varepsilon_n/8$, and we are done. □

Proof of Lemma 3.6. Write $M = (m_{i,j}) = D(I + N)$ with $N = D^{-1}R$. One has:

$$N = (\nu_{i,j}), \quad \text{with} \quad \nu_{i,i} = 0 \quad \text{and} \quad \nu_{i,j} = \frac{m_{i,j}}{m_{i,i}} \text{ for } j \neq i\,. \tag{3.14}$$

We shall show that $\|N\| \leq 1/2$ by using the (unweighted) Schur test, which we recall ([6], Problem 45):

Proposition 3.7 *Let $(a_{i,j})_{1 \leq i,j \leq n}$ be a matrix of complex numbers. Suppose that there exist two positive numbers $\alpha, \beta > 0$ such that:*

1. *$\sum_{j=1}^n |a_{i,j}| \leq \alpha$ for all i;*
2. *$\sum_{i=1}^n |a_{i,j}| \leq \beta$ for all j.*

Then, the (Hilbertian) norm of this matrix satisfies $\|A\| \leq \sqrt{\alpha\beta}$.

It is essential for our purpose to note that:

$$i < j \quad \Longrightarrow \quad |\nu_{i,j}| \leq 32\, \delta^{j-i}\,, \tag{3.15}$$

$$i > j \quad \Longrightarrow \quad |\nu_{i,j}| \leq 32\, (2\, \delta)^{i-j}\,. \tag{3.16}$$

Indeed, we see from (3.11) and (3.14) that, for $i < j$:

$$|\nu_{i,j}| = \frac{|m_{i,j}|}{m_{i,i}} \leq 32\, \varepsilon_i\, \varepsilon_j\, \varepsilon_i^{-2} \delta^{j-i} \leq 32\, \delta^{j-i}$$

since $\varepsilon_j \leq \varepsilon_i$. Secondly, using $\varepsilon_j / \varepsilon_i \leq 2^{i-j}$ for $i > j$ (recall that we assumed that $\varepsilon_{k+1} \geq \varepsilon_k / 2$), as well as $|m_{i,j}| = |m_{j,i}|$, we have, for $i > j$:

$$|\nu_{i,j}| = \frac{|m_{j,i}|}{m_{i,i}} \leq 32\, \frac{\varepsilon_j}{\varepsilon_i}\, \delta^{i-j} \leq 32\, (2\, \delta)^{i-j}.$$

Now, for fixed i, (3.15) gives:

$$\begin{aligned} \sum_{j=1}^{n} |\nu_{i,j}| &= \sum_{j>i} |\nu_{i,j}| + \sum_{j<i} |\nu_{i,j}| \leq 32 \left(\sum_{j>i} \delta^{j-i} + \sum_{j<i} (2\, \delta)^{i-j} \right) \\ &\leq 32 \left(\frac{\delta}{1-\delta} + \frac{2\, \delta}{1 - 2\, \delta} \right) \leq 32\, \frac{3\, \delta}{1 - 2\, \delta} \leq \frac{96}{198} \leq \frac{1}{2}\,, \end{aligned}$$

since $\delta \leq 1/200$. Hence:

$$\sup_i \Big(\sum_j |\nu_{i,j}| \Big) \leq 1/2\,. \tag{3.17}$$

In the same manner, but using (3.16) instead of (3.15), one has:

$$\sup_j \Big(\sum_i |\nu_{i,j}| \Big) \leq 1/2\,. \tag{3.18}$$

Now, (3.17), (3.18) and the Schur criterion recalled above give:

$$\| N \| \leq \sqrt{1/2 \times 1/2} = 1/2\,,$$

as claimed. $\square$

Remark. We could reverse the point of view in the preceding proof: start from θ and see what lower bound for $a_n(C_\varphi)$ emerges. For example, if $\theta(h) \approx h$ as is the case for lens maps (see [11]), we find again that $a_n(C_\varphi) \geq \delta_0 > 0$ and that C_φ is not compact. But if $\theta(h) \approx h^{1+\alpha}$ with $\alpha > 0$, the method only gives $a_n(C_\varphi) \gtrsim \mathrm{e}^{-\alpha n}$ (which is always true: see [11], Theorem 2.1), whereas the methods of [11] easily give $a_n(C_\varphi) \gtrsim \mathrm{e}^{-\alpha \sqrt{n}}$. Therefore, this μ-method seems to be sharp when we are close to non-compactness, and to be beaten by those of [11] for "strongly compact" composition operators.

3.1 Optimality of the EKSY result

El Fallah, Kellay, Shabankhah and Youssfi proved in [5] the following: if φ is a Schur function such that $\varphi \in \mathcal{D}$ and $\|\varphi^p\|_{\mathcal{D}} = O\,(1)$ as $p \to \infty$, then φ is a symbol on $\mathcal{D}$. We have the following theorem, already stated in the Introduction, which shows the optimality of their result.

Theorem 3.8 *Let $(M_p)_{p\geq 1}$ be an arbitrary sequence of positive numbers such that $\lim_{p\to\infty} M_p = \infty$. Then, there exists a Schur function $\varphi \in \mathcal{D}$ such that:*

1) $\|\varphi^p\|_{\mathcal{D}} = O\,(M_p)$ as $p \to \infty$;

2) φ is not a symbol on $\mathcal{D}$.

Remark. We first observe that we cannot replace lim by lim sup in Theorem 3.8. Indeed, since $\varphi \in \mathcal{D}$, the measure $\mu = n_\varphi\, dA$ is finite, and

$$\|\varphi^p\|_{\mathcal{D}}^2 = p^2 \int_{\mathbb{D}} |w|^{2p-2}\, d\mu(w) \geq c\, p^2 \bigg(\int_{\mathbb{D}} |w|^2\, d\mu(w) \bigg)^{p-1} \geq c\, \delta^p\,,$$

where c and δ are positive constants.

Proof of Theorem 3.8. We may, and do, assume that (M_p) is non-decreasing and integer-valued. Let $(l_n)_{n\geq 1}$ be an non-decreasing sequence of positive integers tending to infinity, to be adjusted. Let Ω be the subdomain of the right half-plane $\mathbb{C}_0$ defined as follows. We set:

$$\varepsilon_n = -\log(1 - 2^{-n}) \sim 2^{-n}\,,$$

and we consider the (essentially) disjoint boxes ($k = 0, 1, \ldots$):

$$B_{k,n} = B_{0,n} + 2k\pi i\,,$$

with:

$$B_{0,n} = \{u \in \mathbb{C}\,;\ \varepsilon_{n+1} \leq \Re\mathrm{e}\, u \leq \varepsilon_n \text{ and } |\Im\mathrm{m}\, u| \leq 2^{-n}\pi\}\,,$$

as well as the union

$$T_n = \bigcup_{0<k<l_n} B_{k,2n}\,,$$

which is a kind of broken tower above the "basis" $B_{0,2n}$ of even index.

We also consider, for $1 \leq k \leq l_n - 1$, very thin vertical pipes $P_{k,n}$ connecting $B_{k,2n}$ and $B_{k-1,2n}$, of side lengths 4^{-2n} and $2\pi(1 - 2^{-2n})$ respectively:

$$P_{k,n} = P_{0,n} + 2k\pi i\,,$$

and we set:

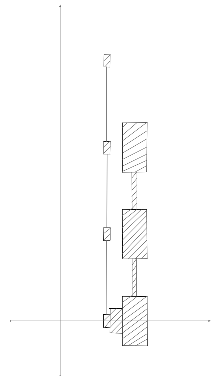
$$P_n = \bigcup_{1\leq k<l_n} P_{k,n}$$

Finally, we set:

$$F = \bigg(\bigcup_{n=2}^{\infty} B_{0,n} \bigg) \cup \bigg(\bigcup_{n=1}^{\infty} T_n \bigg) \cup \bigg(\bigcup_{n=1}^{\infty} P_n \bigg)$$

and:

$$\Omega = \overset{\circ}{F}$$



Then Ω is a simply connected domain. Indeed, it is connected thanks to the $B_{0,n}$ and the P_n, since the $P_{k,n}$ were added to ensure that. Secondly, its unbounded complement is connected as well, since we take one value of n out of two in the union of sets $B_{k,n}$ defining F.

Let now $f\colon \mathbb{D} \to \Omega$ be a Riemann map, and $\varphi = \mathrm{e}^{-f}\colon \mathbb{D} \to \mathbb{D}$.

We introduce the Carleson window $W = W(1, h)$ defined as:

$$W(1,h) = \{z \in \mathbb{D}\,;\ 1 - h \leq |z| < 1 \text{ and } |\arg z| < \pi\, h\}\,.$$

This is a variant of the sets $S(1, h)$ of Section 2. We also introduce the Hastings-Luecking half-windows W'_n defined by:

$$W'_n = \{z \in \mathbb{D}\,;\ 1 - 2^{-n} < |z| < 1 - 2^{-n-1} \text{ and } |\arg z| < \pi\, 2^{-n}\}.$$

We will also need the sets:

$$E_n = \mathrm{e}^{-(T_n \cup B_{0,2n+1} \cup P_n)} = \mathrm{e}^{-(B_{0,2n} \cup B_{0,2n+1} \cup P_{0,n})}\,,$$

for which one has:

$$\varphi(\mathbb{D}) \subseteq \bigcup_{n=1}^{\infty} E_n\,.$$

Next, we consider the measure $\mu = n_\varphi\, dA$, and a Carleson window $W = W(1, h)$ with $h = 2^{-2N}$. We observe that $W'_{2N} \subseteq W$ and claim that:

Lemma 3.9 *One has:*

1) $w \in W'_{2N} \quad \Longrightarrow \quad n_\varphi(w) \geq l_N$;

2) $\|\varphi^p\|^2_{\mathcal{D}} \lesssim p^2 \sum_{n=1}^{\infty} l_n\, 16^{-n}\, \mathrm{e}^{-p\, 4^{-n}}$.

Proof of Lemma 3.9. 1) Let $w = r\,\mathrm{e}^{i\theta} \in W'_{2N}$ with $1-2^{-2N} < r < 1-2^{-2N-1}$ and $|\theta| < \pi\, 2^{-2N}$. As $-(\log r + i\theta) \in B_{0,2N}$, one has $-(\log r + i\theta) = f(z_0)$ for some $z_0 \in \mathbb{D}$. Similarly, $-(\log r + i\theta) + 2k\pi i$, for $1 \leq k < l_N$, belongs to $B_{k,2N}$ and can be written as $f(z_k)$, with $z_k \in \mathbb{D}$. The z_k's, $0 \leq k < l_N$, are distinct and satisfy $\varphi(z_k) = \mathrm{e}^{-f(z_k)} = \mathrm{e}^{-f(z_0)} = w$ for $0 \leq k < l_N$, thanks to the $2\pi i$-periodicity of the exponential function.

2) We have $A(E_n) \lesssim \mathrm{e}^{-2\varepsilon_{2n+2}} 4^{-2n} \leq 4^{-2n}$ (the term $\mathrm{e}^{-2\varepsilon_{2n+2}}$ coming from the Jacobian of e^{-z}) and we observe that

$$w \in E_n \quad \Longrightarrow \quad |w|^{2p-2} \leq (1 - 2^{-2n-1})^{2p-2} \lesssim \mathrm{e}^{-p\,4^{-n}}\,.$$

It is easy to see that $n_\varphi(w) \leq l_n$ for $w \in E_n$; thus we obtain, forgetting the constant term $|\varphi(0)|^{2p} \leq 1$, using (2.5) and keeping in mind the fact that $n_\varphi(w) = 0$ for $w \notin \varphi(\mathbb{D})$:

$$\begin{aligned}
\|\varphi^p\|_{\mathcal{D}}^2 &= p^2 \int_{\varphi(\mathbb{D})} |w|^{2p-2}\, n_\varphi(w)\, dA(w) \\
&\leq p^2 \bigg(\sum_{n=1}^{\infty} \int_{E_n} |w|^{2p-2}\, n_\varphi(w)\, dA(w) \bigg) \\
&\leq p^2 \bigg(\sum_{n=1}^{\infty} \int_{E_n} |w|^{2p-2}\, l_n\, dA(w) \bigg) \\
&\lesssim p^2 \sum_{n=1}^{\infty} l_n\, 16^{-n}\, \mathrm{e}^{-p\,4^{-n}}\,,
\end{aligned}$$

ending the proof of Lemma 3.9. $\square$

End of the proof of Theorem 3.8. Note that, as a consequence of the first part of the proof of Lemma 3.9, one has

$$\mu(W) \geq \mu(W'_{2N}) = \int_{W'_{2N}} n_\varphi\, dA \geq l_N A(W'_{2N}) \gtrsim l_N h^2\,,$$

which implies that $\sup_{0<h<1} h^{-2} \mu[W(1,h)] = +\infty$ and shows that C_φ is not bounded on $\mathcal{D}$ by Zorboska's criterion ([17], Theorem 1), recalled in (2.7).

It remains now to show that we can adjust the non-decreasing sequence of integers (l_n) so as to have $\|\varphi^p\|_{\mathcal{D}} = O\,(M_p)$. To this effect, we first observe that, if one sets $F(x) = x^2\,\mathrm{e}^{-x}$, we have:

$$p^2 \sum_{n=1}^{\infty} 16^{-n}\, \mathrm{e}^{-p\,4^{-n}} = \sum_{n=1}^{\infty} F\left(\frac{p}{4^n}\right) \lesssim 1\,.$$

Indeed, let s be the integer such that $4^s \leq p < 4^{s+1}$. We have:

$$\sum_{n=1}^{\infty} F\left(\frac{p}{4^n}\right) \lesssim \sum_{n=1}^{s} \frac{4^n}{p} + \sum_{n>s} F(4^{-(n-s-1)}) \lesssim 1 + \sum_{n=0}^{\infty} F(4^{-n}) < \infty\,,$$

where we used that F is increasing on $(0,1)$ and satisfies $F(x) \lesssim \min(x^2, 1/x)$ for $x > 0$. We finally choose the non-decreasing sequence (l_n) of integers as:

$$l_n = \min(n, M_n^2)\,.$$

In view of Lemma 3.9 and of the previous observation, we obtain:

$$\begin{aligned}
\|\varphi^p\|_{\mathcal{D}}^2 &\lesssim p^2 \sum_{n=1}^{\infty} 16^{-n}\, \mathrm{e}^{-p\,4^{-n}}\, l_n \\
&\leq p^2 \sum_{n=1}^{p} 16^{-n}\, \mathrm{e}^{-p\,4^{-n}}\, l_p + p^2 \sum_{n>p} 16^{-n} l_n \\
&\lesssim l_p + p^2 \sum_{n>p} 4^{-n} \lesssim l_p + p^2\, 4^{-p} \lesssim M_p^2\,,
\end{aligned}$$

as desired. This choice of (l_n) gives us an unbounded composition operator on $\mathcal{D}$ such that $\|\varphi^p\|_{\mathcal{D}} = O\,(M_p)$, which ends the proof of Theorem 3.8. □

References

[1] N. Arcozzi, R. Rochberg, E. T. Sawyer and B. D. Wick, The Dirichlet space: a survey, *New York J. Math.* 17A (2011), 45–86.

[2] B. Carl and I. Stephani, Entropy, Compactness and the Approximation of Operators, *Cambridge Tracts in Mathematics*, vol. 98 (1990).

[3] T. Carroll and C. Cowen, Compact composition operators not in the Schatten classes, *J. Oper. Theory,* no. 26 (1991), 109–120.

[4] C. Cowen and B. MacCluer, Composition operators on spaces of analytic functions, *CRC Press* (1994).

[5] O. El-Fallah, K. Kellay, M. Shabankhah and H. Youssfi, Level sets and composition operators on the Dirichlet space, *J. Funct. Anal.* 260, no. 6 (2011), 1721–1733.

[6] P. Halmos, A Hilbert space problem book, Second Edition, *Graduate Texts in Mathematics* 19, Springer-Verlag (1982).

[7] W. H. Hastings, A Carleson theorem for Bergman spaces, *Proc. Amer. Math. Soc.* 52 (1975), 237–241.

[8] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Compact composition operators on Bergman-Orlicz spaces, *Trans. Amer. Math. Soc.* 365, no. 8 (2013), 3943–3970.

[9] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Some new properties of composition operators associated to lens maps, *Israel J. Math.* 195, no. 2 (2013), 801–824.

[10] P. Lefèvre, D. Li, H. Queffélec and L. Rodríguez-Piazza, Compact composition operators on the Dirichlet space and capacity of sets of contact points, *J. Funct. Anal.* 264 (2013), no. 4, 895–919.

[11] P. Lefèvre, D. Li, H. Queffélec, L. Rodríguez-Piazza, Approximation numbers of composition operators on the Dirichlet space, *Arkiv för Mat.* doi. 10.1007/s11512-013-0194-z

[12] D. Li, H. Queffélec and L. Rodríguez-Piazza, On approximation numbers of composition operators, *J. Approx. Theory* 164, no. 4 (2012), 431–459.

[13] D. Li, H. Queffélec and L. Rodríguez-Piazza, Estimates for approximation numbers of some classes of composition operators on the Hardy space, *Ann. Acad. Sci. Fenn. Math.*, Vol. 38 (2013), 1–18.

[14] B. MacCluer and J. Shapiro, Angular derivatives and compact composition operators on the Hardy and Bergman spaces, *Canad. J. Math.* 38, no. 4 (1986), 878–906.

[15] A. Pietsch, s-numbers of operators in Banach spaces, *Studia Math.* LI (1974), 201–223.

[16] W. T. Ross, The classical Dirichlet space, Recent advances in operator-related function theory, 171–197, *Contemp. Math.* 393, Amer. Math. Soc., Providence, RI (2006).

[17] N. Zorboska, Composition operators on weighted Dirichlet spaces, *Proc. Amer. Math. Soc.* 126, no. 7 (1998), 2013–2023.

Daniel Li, Univ Lille Nord de France,
U-Artois, Laboratoire de Mathématiques de Lens EA 2462
& Fédération CNRS Nord-Pas-de-Calais FR 2956,
Faculté des Sciences Jean Perrin, Rue Jean Souvraz, S.P. 18,
F-62 300 LENS, FRANCE
daniel.li@euler.univ-artois.fr

Hervé Queffélec, Univ Lille Nord de France,
USTL, Laboratoire Paul Painlevé U.M.R. CNRS 8524 & Fédération CNRS Nord-Pas-de-Calais FR 2956,
F-59 655 VILLENEUVE D'ASCQ Cedex, FRANCE
Herve.Queffelec@univ-lille1.fr

Luis Rodríguez-Piazza, Universidad de Sevilla,
Facultad de Matemáticas, Departamento de Análisis Matemático & IMUS,
Apartado de Correos 1160,
41 080 SEVILLA, SPAIN
piazza@us.es